By the AM-GM Inequality, we have

$$\prod_{\text{cyclic}} (a+\sqrt{ab}) = \sqrt{abc} \prod_{\text{cyclic}} (\sqrt{a}+\sqrt{b})$$
$$\geq 8\sqrt{abc}\sqrt{\sqrt{a}\sqrt{b}\sqrt{b}\sqrt{c}\sqrt{c}\sqrt{a}} = 8abc,$$

so our proof is complete. Clearly, equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3574. [2010 : 398, 400, 548, 550] *Proposed by Michel Bataille, Rouen, France.*

Let x, y, and z be real numbers such that $x+y+z=0$. Prove that

$$\sum_{\text{cyclic}} \cosh x \;\leq\; \sum_{\text{cyclic}} \cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1 + 2\sum_{\text{cyclic}} \cosh x\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $a := e^x, b = e^y, c = e^z$. Then $a, b, c > 0$ and $abc = e^{x+y+z} = 1$. Let $s := a+b+c, p := ab+ac+bc$. Then

$$\sum_{cyc} \cosh(x) = \frac{1}{2}\sum_{cyc}(a+bc) = \frac{s+p}{2}\,.$$

Let's observe that

$$\cosh\left(\frac{x-y}{2}\right) = \frac{e^{\frac{x-y}{2}} + e^{\frac{y-x}{2}}}{2} = \frac{1}{2}\left(\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}\right)$$
$$= \frac{a+b}{2\sqrt{ab}} = \frac{(a+b)\sqrt{c}}{2}\,. \tag{1}$$

Thus

$$\sum_{cyc} \cosh^2\left(\frac{x-y}{2}\right) = \frac{1}{4}\sum_{cyc}(a+b)^2 c = \frac{1}{4}\sum_{cyc} a^2c + b^2c + 2 = \frac{3+sp}{4}\,.$$

Also

$$\prod_{cyc} \cosh(x) = \prod_{cyc} \frac{a+bc}{2} = \prod_{cyc} \frac{a^2+1}{2a}$$
$$= \frac{1}{8}\prod_{cyc}(a^2+1) = \frac{2+p^2+s^2-2p-2s}{8}\,.$$

Thus, the inequality to prove becomes

$$\frac{1}{2}(s+p) \leq \frac{3+sp}{4} \leq 1 + \frac{2+p^2+s^2-2p-2s}{4}\,, \qquad (2)$$

or equivalently

$$2(s+p) \leq 3+sp \leq 6+p^2+s^2-2(p+s)\,. \qquad (3)$$

Observing that $p \geq 3\sqrt[3]{a^2b^2c^2} = 3$ and $s \geq 3\sqrt[3]{abc} = 3$ we obtain

$$sp + 3 - 2(s+p) = (s-3)(p-3) + (s-3) + (p-3) \geq 0\,,$$

which proves the left hand side of (3).

To prove the RHS of (3) we note that

$$\begin{aligned} 6+&p^2+s^2-2(p+s)-(3+sp) \\ &= 3+p^2+s^2-2p-2s-sp = (p-s)^2+sp+3-2(s+p) \qquad (4)\\ &= (p-s)^2+(s-3)(p-3)+(s-3)+(p-3) \geq 0\,. \end{aligned}$$

This proves the RHS of (3), and thus completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3575. [2010 : 398, 400] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with incentre I. Characterize the lines through I intersecting the sides AB and AC at D and E, respectively, such that $DE = DB + EC$ and determine how many such lines there are in terms of $\angle B$ and $\angle C$.

Solution by Joel Schlosberg, Bayside, NY, USA.

For each line DE through I intersecting sides AB and AC, $\angle BID$ and $\angle CIE$ are external to ΔBIC; moreover, because $\angle BID + \angle CIE = 180^\circ - \angle BIC = \frac{B+C}{2}$, each position of DE corresponds to a unique angle $\theta \in [-\frac{B}{4}, \frac{C}{4}] \subset [-45^\circ, 45^\circ]$ such that

$$\angle BID = \frac{B}{2} + 2\theta \quad \text{and} \quad \angle CIE = \frac{C}{2} - 2\theta.$$